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# MATHEMATICAL TROUBLES OF THE FRESHMAN[1]

By G. A. MILLER, University of Illinois.

The fact that there is generally a great gap between high-school mathematics and that of the freshman courses in our colleges and universities accounts for a considerable part of the troubles of the first year students in the higher institutions of learning. While it may not afford us any comfort to know that similar troubles exist in other countries, it is evidently desirable to acquaint ourselves with the extent of these troubles before we endeavor to explain their source or to devise means to mitigate them.

**Conditions in Germany.**—In the preface of his interesting book entitled *Die Erziehung der Anschauung*[2] Professor H. E. Timerding observes, in reference to conditions in Germany, that a deep gap divides the whole system of education. On the one side are the higher institutions of learning while on the other are the elementary schools. Everything is different on these two sides,—the objects, the methods, the views, the text-books; they are two completely separated worlds. The one has its center in scientific research, the other in practical education, and this separation is so complete that the results of scientific research come to the elementary teachers only through one or two intermediaries and in a very misleading form; while on the other hand, the university teachers pass by all pedagogic questions with great disdain.

**Conditions in America.**—Our American university student has perhaps one advantage over his German brother as a result of the fact that this one deep gap in the German system is generally replaced by two somewhat less serious ones in our system of mathematical education. One of these gaps is met on entering the freshman courses while the other is reached a few years later in the first courses of a really graduate nature. As the latter of these is encountered by a comparatively small number, whose deep interest inspires renewed courage, the former presents by far the more serious question to those interested in the

[1] Read before the Mathematics Section of the California High School Teachers Association, July 8, 1913.

[2] Published by B. G. Teubner in 1912.

welfare of our mathematical education. We shall therefore turn our entire attention to the gap at the beginning of the freshmen courses.

**Some of the Causes.**—It is not difficult to find some of the forces tending to widen this gap or to reopen it, if it should have been closed locally through heroic efforts. We have only to look at the company which our college and university teachers keep. They like to read books on higher mathematics and to investigate new or only partly explored mathematical regions. They like to go to meetings of mathematical investigators and to join in a language which is entirely unintelligible to the outside world. They love to go abroad to the biggest centers of mathematical lore and to expose themselves to the zeal for great new ideas whose enchanting comprehensiveness makes pigmies of the mathematical concepts which bewilder the freshman and strain his powers of comprehension.

Is it a wonder that such men as these should recoil from contact with those who are mainly concerned with the difficult problem of enlarging the horizon of young minds whose penetrating powers, though great when compared with those of the uneducated man, are trivial when compared with those which may be attained by many years of arduous application. The tastes of the college and university instructor, as indicated by the company which these men keep, cannot fail to call forth most serious misgivings on the part of those who are interested in a gradual and continuous mathematical development of the student.

**The Widening Gap.**—An important element in this situation is the fact that some of the influences tending to increase the difficulties with which the freshman has to contend are becoming more serious from year to year. With the increasing emphasis on graduate courses there comes a growing interest in research work, and a university instructor is often led to feel that his advancement in standing both in the university and among his colleagues is mainly based upon his activity in research. It is only reasonable to expect that such instructors should seek to affiliate themselves with the agencies which tend to promote their own advancement and to secure the approbation of their colleagues.

In view of these gaps in our system of mathematical education and the strong forces which tend to perpetuate them, we may at first be inclined to turn our attention to the means of bridging them at any cost. Conferences like this between high school and university instructors are steps along this line. The great work which has been accomplished during the last few years under the general direction of the International Commission on the Teaching of Mathematics tends to bring high school and university teachers in closer touch, and hence it helps to bridge the serious gap met by the freshman. However much good these efforts may accomplish they cannot eliminate permanently conditions arising from the nature of our educational system. There are cases where the complete cure of an evil is more objectionable than the evil itself, and the conditions under consideration may possibly belong to this category. At any rate, it seems wise to consider some of the effects of the gaps in our system of mathematical education before prescribing very drastic measures for closing them, or even for bridging them in many places.

**Our Patch-work System.**—In most schools we begin our mathematical education by leaving great gaps in the subject matter. For instance, algebra and geometry are developed as distinct subjects and the gap between them is gradually filled up as we proceed. If it is found feasible to start our mathematical farm by the cultivation of isolated patches, it may possibly be found feasible to have isolated patches in our methods of work and in our centers of interest. It may be that at the time the student reaches the university he ought to get into a new world with new views, new objects, new perspectives, and new mathematical ideals.

We are sometimes told that our first year of university instruction is bad because those who stood the highest in their classes in the high school do not make as good freshman records as some of those who did only mediocre work in the high school. Taken by itself this does not prove anything. The mediocre student in the high school may appear mediocre only in view of some arbitrary and unjust standards, or it may be that these standards are just at the high school age but would not be just later in life. The qualities leading to distinction in youth do not always lead to equal distinction in manhood. Adaptation is one of the most important faculties to cultivate, and if the student learns, at the beginning of his college course, the hard lesson that he has sometimes to adapt himself to new and radically different conditions, he has learned one of the most important lessons of life. For instance, young people do not grow gradually into the married state but they are required to adapt themselves at one step. Discontinuity is common in life and we may naturally expect to meet it in education.

**Inherent Difference.**—Before trying to determine whether the extensive freshman fatalities are due to high school or to college methods of instruction let us agree that these two methods ought to be different. After agreeing on this point we still have the main difficulty to solve; namely, how great this difference should be. On this point, we can probably not expect to reach perfect agreement. The more advanced the work is the more the individuality of the teacher does and ought to assert itself. Hence we should not expect as much uniformity among the methods used by freshman instructors as among those employed by the high school teachers. At this point we reach again the perplexing question how great this difference ought to be.

While my arguments thus far have been directed mainly towards placing the freshman situation in a charitable light, I would not desire to be classed with those who are entirely satisfied with the situation and who see no room for improvement. Without trying to turn high school teachers into university teachers or vice versa, I would urge most strongly that these two classes of teachers should regard it as one of their most important duties to learn from each other. The tendency to grow away from each other is exceedingly strong, and a conscious and repeated effort to overcome this tendency is necessary, in most cases, in order to maintain ourselves in a condition to render the best service. There is much that we can learn from each other.

**Higher Ideals Needed.**—The high school teachers, as a class, need higher scholarly ideals. The high school teachers' meetings should have more papers dealing with modern mathematical subjects, and the high school teachers' library should contain more books and journals dealing with advanced mathematics. A teacher who has ceased to hunger for more knowledge about his subject has no business in the high school. Let us not deceive ourselves. Our students know where our interests lie, and how can we expect to inspire them with zeal in the subject matter if we have not sufficient interest in it to work ahead to the best of our abilities and in accordance with our opportunities.

**Mathematical Journals.**—One of the chief incentives to come to a meeting of this kind should be the opportunity it gives us to learn, by consultation with each other or by means of the excellent library at this place, more about some mathematical questions. The mathematical journals should be one of the chief centers of interest, since they are of such varied grades as to come within the reach of all. At this time high school teachers should have especial interest in the improvement of the AMERICAN MATHEMATICAL MONTHLY, which is peculiarly their own journal in this country, from the present point of view. The *Mathematical Gazette* of England has somewhat similar aims.

Those who read foreign languages and who desire a journal of about the same grade in such a language, would probably find the following (in French, German, Italian, and Spanish, respectively) very instructive: *L'Enseignement Mathématique, Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, Periodico di Matematica,* and *Revista de la Sociedad Matemática Española.* Some of the other foreign languages which are less extensively read by Americans have secondary mathematical journals. For instance, Japan has such a journal which is known as "The XY" and is published in the Japanese language by the XY House. A brief review of this journal is found in the *Mathematical Gazette* for January, 1913, page 17.

**Interest in Pedagogical Questions.**—While the high school teacher can do much towards decreasing the gap at the beginning of the freshman year by taking a deeper interest in the subject matter and in the fields which lie just beyond those with which he is primarily concerned, the university instructor of freshman mathematics should not forget that the strong temptations to grow away from pedagogic interest must be partially counteracted. Some of our European colleagues are setting us conspicuous examples along this line. The fact that the office of President of the Mathematical Association of England, an association of teachers and students of elementary mathematics, has been filled in recent years by such eminent men as A. R. Forsyth, B. G. Mathews, G. H. Bryan, H. H. Turner, E. W. Hobson and Sir George Greenhill, illustrates the attitude of some of the most prominent English mathematicians with respect to the work of the secondary teachers.

A few of the leading American mathematicians are also taking a deep interest in high school mathematics, but I do not know of any American association of teachers of elementary or secondary mathematics which can point to a list of

such eminent successive presidents as the one which has just been cited. Whether this is due to a lack of appreciation of scientific eminence on the part of our teachers of secondary mathematics, or whether it is mainly due to the fact that too few eminent American mathematicians are interested in elementary and in secondary mathematics, may be difficult to determine. Probably both of these undesirable conditions have their influence and tend to strengthen each other. At any rate, our leading mathematicians and our teachers of secondary mathematics should be in closer touch with each other.

**Great Learning and Simple Language.**—The teachers of elementary mathematics are sometimes afraid of the university teachers because the latter use a language of their own. This is too often true, but it is not true of all the really great men. Great *learning* tends towards a simple language while great *cramming* tends toward an unintelligible jargon. If a man cannot talk so as to be understood you can be almost sure that he has read much but understood little, or that he has dreamed himself into many problems but did not take the time to solve any of them completely. Such dreamers may be helpful to the research student but they have little for the busy teacher.

On the other hand, the great mathematician who has really mastered many of the problems of advanced mathematics should be able to talk most clearly about the difficult mathematical questions which confront the teacher of secondary mathematics. Hence these teachers should seek great mathematicians as presidents of their associations in order that they may be able to learn new things easily at the meetings, and thus be enabled to present subjects more clearly to their students.

Among the subjects which are at the present time of common interest to many advanced mathematicians and to high school teachers, the foundations of geometry and the theories of aggregates and groups deserve especial mention.

**Special Class Preparation Needed.**—The instructor of the freshmen students of our colleges probably fails more frequently on account of a lack of a complete mastery of the subject which he is to teach than on account of the fact that his methods differ from those of the high school. The student expects and welcomes a change of method. He likes to get into a new atmosphere. He welcomes the greater freedom and the greater individuality of methods which he finds in college. The great disturbing element often is a lack of clearness and simplicity. This may be due to the fact that his instructor was thinking along an entirely different line until five minutes before the class. What we need most of all is a deeper interest in the subject matter on the part of our teachers, and when our teachers' conferences come to be dominated more completely by this thought, our university instructors and high school teachers will have more in common, and the gap at the beginning of the university course will become the freshman's delight instead of his stumbling block.

**Resumé.**—The substance of our argument is, in brief, as follows: The scientific interests of the high school teacher and the university instructor should have more in common. As the latter have generally better opportunities and more

time for study, the former should more frequently invite the latter to assist them in the solution of their more difficult scientific questions. On the other hand, the university instructor should realize that his freshman classes deal with subjects which he himself understands imperfectly, and he should be interested in arriving continually at clearer views in regard to these subjects and the methods of presenting them.

**The Investigator and the Teacher.**—In recent years there has been considerable discussion in reference to making a public distinction between the investigator and the teacher on the university faculty. Full professors of mathematics have been appointed, in excellent universities, mainly on the ground of their success as teachers. The question might be raised whether it would not be wiser to insist that every successful university instructor must be an investigator, but at the same time to admit that investigation in *elementary* subjects is also important, and especially in respect to the teaching of these subjects. Those who are investigating in the fields of freshman and high school mathematics should, and in many cases now actually do, fill the so-called teaching professorships.

**Mathematics a Growing Subject.**—One of the most harmful phrases is the assertion that "mathematics is a finished science." It would be difficult to find a shorter way to expose dense ignorance of the subject than to use this phrase seriously. Even elementary mathematics is not a finished science and indeed no one has a complete knowledge of any elementary subject.

The feeling that we are all dealing with a growing subject and that new developments in one branch are apt to throw light on other branches should awaken in us a hopeful interest in the work in the different branches. Such an interest tends to inspire us with a zeal that is contagious. This zeal is one of the greatest assets of any teacher of mathematics. It serves also as a bond which unites us all and increases our respect for the work in the various fields of mathematics. The university instructor who has great respect for his high school colleague is almost sure to be a good teacher, and the high school teacher who honors the work which his university colleague tries to do is apt to continue to grow and to inspire his students with his own zeal.

**The Beauties of Mathematics.**—We are all inspired by the beauties of this country when viewed from the tops of the various hills and mountains of this state. The views from this campus enthuse us and lead us to express our delight to those around us. The descriptions of the beauty of the "Golden Gate" have aroused most interesting impressions in the minds of millions of school children, as well as in the minds of adults who have never visited this part of the world. Is it not a shame that the intellectual beauties seen from intellectual heights should be passed by without also awakening our emotions? Elementary algebra and elementary geometry are largely blooming orchards. The fruit will appear later. Let us not talk continually about the fruit now. The tendency of the modern text-book is to disfigure a beautiful blooming tree by tying pictures of the fruit on its branches. I believe that more students can be inspired by the blossoms which they see than by the pictures of the fruit. At any rate, let us be

alive to the beauties of our subjects, no matter how elementary. In this way most freshman mathematical troubles will vanish notwithstanding the change of emphasis.

Let me carry this figure a little further. What I mean is that the high school teacher should have traveled widely through the beauties of mathematics and should not hesitate to talk about some of the beauties which are beyond his students. We do not hesitate to talk about the Grand Canyon of the Colorado or about the grandeur of Yosemite to those who never expect to visit these places. Why should we then hesitate to mention some of the jewels of the calculus, the grandeur of number realms, or the wonders of higher dimensions, even to those who are not sure that they will ever reach these stages. In most cases the answer is evident. We know too little ourselves about matters lying close to the subjects we teach, and therefore we cannot talk about these things in a sufficiently simple language. We have not seen the subjects ourselves but merely read about them. Teaching the most advanced subject which we have studied is like describing a country about which we have read but which we have not seen.

**The Chief Essential of Good Teaching.**—If I were asked to provide a formula for the manufacture of successful high school and university teachers, I should consider it necessary to include a great number of ingredients. Hence my emphasis on more scholarship *relating directly to the subject in hand* means simply that from my point of view the supply of this ingredient is, at the present time, more commonly inadequate than that of any other. If we could add a sufficiently large amount of this, together with the zeal which naturally goes with it, to the present equipment of our teachers, most of the present freshman troubles would vanish. Possibly the great prominence of athletic matters as chief subjects of conversation might even become a thing of the past.

**Journals and Books.**—While it is expensive to study at a university or to attend a large number of meetings, it is comparatively inexpensive to secure a few good journals and good books. Most of the best thoughts of the world find their way into journals and then into books,[1] and the most economical way of thinking is to relate ourselves to these fresh thoughts. Hence the need of good libraries is being recognized as never before, and the teacher's private library should be as complete as possible along the lines in which his work lies. As his subject connects with other realms of growing knowledge, such a library will enable him to relate his life with the sympathies and interest of all seekers of truth. This vital connection ennobles our lives as teachers and furnishes the most important element of that inspiration and sympathy between teacher and student which surmounts all obstacles.

**Journals as Antidotes.**—The rapid increase in the number and types of mathematical journals during the last fifty years reflects their great value. While Americans are generally quick to adopt the best, the American teachers of mathematics have been very slow in improving their efficiency by reading the mathe-

[1] Cajori's history of logarithms which has been appearing in the MONTHLY is now in the journal stage.

matical journals. The reading of the same things goes far toward creating points of contact and developing common interests, and it is these common interests between high school and university instructors which will tend to remove the objectionable features of the gap with which the freshman has to contend. When an evil in the physical world begins to assume serious proportions, its antidotes become prominent objects of study. One of the strongest antidotes against the forces which tend to produce harmful gaps in our mathematical education may be found in the uniting tendencies of the mathematical journals. As these journals constitute such a strong directing and inspiring force and provide a forum where all may meet at will for the discussion of problems, they should be heartily supported by all teachers of mathematics.

In closing I desire to lay great stress on another point, namely the history of the subjects we teach. The normal student is most deeply interested in some historical facts but most of us are too lazy or too indifferent to study the history of our subject sufficiently to speak about it in simple and accurate language. A few historical observations now and then put new life into our classes. It is all the more necessary that we cultivate a continuous acquaintance with the history of our subjects since the recent progress of our knowledge along this line has been so rapid. In history we find a bond which unites not only all those interested in mathematics but also all who are interested in the development of the intellect.

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## ON THE SOLUTIONS OF LINEAR EQUATIONS HAVING SMALL DETERMINANTS.

By F. R. MOULTON, University of Chicago.

The fundamental theorem respecting linear non-homogenous equations is that their solution exists and is unique if the determinant of the coefficients of the undetermined quantities is distinct from zero. But this simple theorem does not by any means contain all that is important from a practical point of view. If the coefficients of the equations are furnished by observations, or by measurements, they are determined only to a certain number of places. The question then arises how exact their solution is. In the discussion of this question the magnitude of the determinant plays an important rôle, and the results are particularly interesting and striking when the determinant is small.

For the sake of illustration of some of the peculiarities of the solutions of linear systems, before taking up the general discussion, consider the equations

$$(1) \qquad \begin{cases} .34622\,x + .35381\,y + .36518\,z = .24561, \\ .89318\,x + .90274\,y + .91143\,z = .62433, \\ .22431\,x + .23642\,y + .24375\,z = .17145. \end{cases}$$